Limit laws for stochastic cumulative transition matrices

Let $\mathcal{X} = \left(X(t)\right)_{t=1,\ldots,T}$ denote a discrete Markov chain of $N \times N$ complex matrices, where $X(1), \ldots, X(T)$ are not necessarily independent or identically distributed. What can we say about the distribution of the eigenvectors and eigenvalues of $Y(T) = \prod_{t=1,\ldots,T} X(t)$? For example, under what conditions can we expect a Central Limit Theorem-type result to hold for $\frac{1}{T}\log Y(T)$ as $T \to \infty$?

The question is motivated by the need to understand the distribution of

- cumulative transition matrices in credit models where the one-period stochastic transition matrices are serially correlated;
- cumulative conditional transition matrices where the conditioning variables are stochastic and serially correlated.

There is vast literature on this problem interpreted as a dynamical system on $N \times N$, and the analogues of the Central Limit Theorem are obtained for the normalized *entries* of $Y(T)$ under appropriate assumptions, like $X(t)$ having strictly positive entries; see e.g. [1] and [2]. The appropriate form of the Central Limit Theorem is also obtained for the case where the terms of the product are perturbations of the identity, like $Y(T) = \prod_{t=1,\ldots,T}\left(I + \frac{X(t)}{T}\right)$ or $Y(T) = \prod_{t=1,\ldots,T}\left(I + \frac{X(t)}{\sqrt{T}}\right)$; see e.g. [3] and [4]. Such results seem to be unrelated to the problem stated above.

During the triage workshop, I will present examples showing that a Central Limit Theorem-type result does not hold for $\frac{1}{T}\log Y(T)$ in general, which may help to identify the assumptions under which positive results can be obtained.

[1] R. Cavazos-Cadena, D. Hernández-Hernández: A central limit theorem for normalized products of random matrices. Periodica Mathematica Hungarica Vol. 56 (2), 2008, pp. 183-211. DOI: 10.1007/s10998-008-6183-3.

[2] H. Furstenberg, H. Kesten: Products of Random Matrices. Ann. Math. Statist. 31(2): 457-469 (1960). DOI: 10.1214/aoms/1177705909.

[3] D. Huang, J. Niles-Weed, J. A. Tropp, R. Ward: Matrix Concentration for Products. Foundations of Computational Mathematics (2022) 22, 1767-1799. https://doi.org/10.1007/s10208-021-09533-9.

[4] C. Sadel, B. Virág: A Central Limit Theorem for Products of Random Matrices and GOE Statistics for the Anderson Model on Long Boxes. Commun. Math. Phys. 343, 881-919 (2016). DOI: 10.1007/s00220-016-2600-4.

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